Copyright © 2005 The ANSOL Corporation

WHY IT IS NECESSARY TO USE DATA FROM MORE THAN ONE STRAIN FIELD IN DETERMINING THE HELMHOLTZ FREE-ENERGY (STRAIN ENERGY) DENSITY FUNCTION

By

Kenneth N. Morman, PhD
The ANSOL Corporation
West Bloomfield, MI. 48323

Background

The constitutive theory of *hyperelasticity* (Rivlin, 1948) is often employed to represent the stress-strain response behavior of incompressible, isotropic rubber-like solids. The theory relates the three-dimensional *Cauchy* (true) *stress tensor* **T** to the derivatives of the Helmholtz free-energy (strain energy) density function Ψ by:

$$\mathbf{T} = -p\mathbf{I} + 2\left(\partial\Psi/\partial I_1 + I_1\partial\Psi/\partial I_2\right)\mathbf{B} - 2\partial\Psi/\partial I_2\mathbf{B}^2 \tag{1}$$

where: **I** is the *identity tensor*; **B** is the *left Cauchy-Green deformation tensor* characterized by the three invariants

$$I_1 = tr\mathbf{B},\ I_2 = \tfrac{1}{2}\left(I_1^2 - tr\mathbf{B}^2\right),\ I_3 = \det\mathbf{B}; \tag{2}$$

and p is a Lagrange multiplier which arises due to the material incompressibility constraint $I_3 = 1$.

Herein we use the terms Helmholtz free-energy density and strain energy density interchangeably. Strictly speaking, the terms Helmholtz free-energy density and strain energy density are used for non-isothermal and isothermal processes, respectively.

For homogeneous deformations, eqn (1) can be expressed in principal component form by imposition of the relations $T_{12} = T_{23} = T_{13} = 0$ and $t_1 = T_{11}, t_2 = T_{22}, t_3 = T_{33}$:

$$t_i = -p + 2\left[\partial\Psi/\partial I_1 + I_1\partial\Psi/\partial I_2\right]\lambda_i^2 - 2\left(\partial\Psi/\partial I_2\right)\lambda_i^4, \quad i = 1,2,3 \tag{3}$$

A *simple extension* is defined by stretch ratios: $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda_1}$. If we assert that no stresses are applied in the 2 and 3 directions, then $t_2 = t_3 = 0$, and eqn (3) for t_2 and t_3 becomes an equation for the unknown Lagrange multiplier p:

$$p = 2\left[\partial\Psi/\partial I_1 + I_1\partial\Psi/\partial I_2\right]\lambda_2^2 - 2\left(\partial\Psi/\partial I_2\right)\lambda_2^4 \quad (4)$$

When this result is inserted in eqn (3) for t_1, the result is

$$t_1 = 2\left[\partial\Psi/\partial I_1 + I_1\partial\Psi/\partial I_2\right]\left(\lambda_1^2 - \lambda_2^2\right) - 2\left(\partial\Psi/\partial I_2\right)\left(\lambda_1^4 - \lambda_2^4\right) \quad (5)$$

This equation gives the stress-strain relation in tension in terms of two properties of the material, $\partial\Psi/\partial I_1$ and $\partial\Psi/\partial I_2$. In the present case, the three invariants are given by $I_1 = \lambda_1^2 + 2\lambda_2^2$, $I_2 = 2\lambda_1^2\lambda_2^2 + \lambda_2^4$ and $I_3 = \lambda_1\lambda_2\lambda_3 = 1$, or $\lambda_2 = \lambda_3 = 1/\sqrt{\lambda_1}$.

Experimental Evaluation of the Helmholtz Free-energy (Strain Energy) Density Function Ψ

In order to determine the form of $\Psi(I_1, I_2)$ from experimental data, it is necessary to make measurements of stress-strain relations under *different* types of strain field. Obata *et. al.* (1970) discovered in their attempts to construct $\partial\Psi(I_1, I_2)/\partial I_1$ and $\partial\Psi(I_1, I_2)/\partial I_2$ surfaces for natural rubber vulcanizates that the whole region of the (I_1, I_2) domain for homogeneous deformations under the assumptions of material incompressibility can be mapped out from a region of stretch ratios λ_1, λ_2 satisfying the condition $\lambda_2 \le \lambda_1$ and $\lambda_2 \ge 1/\lambda_1^{1/2}$ (Fig. 1). Because of this interrelationship between the invariants I_1 and I_2 for simple types of strain, Treloar (1975, p. 218) cautions against the use of experimental data from any one particular type of strain field, e.g. simple extension or pure shear for deriving the true form of $\Psi(I_1, I_2)$. It can be seen in Fig. 2 any such simple strain only traces out a single line on the free-energy surface and does not provide sufficient information for constructing the surface.

Copyright © 2005 The ANSOL Corporation

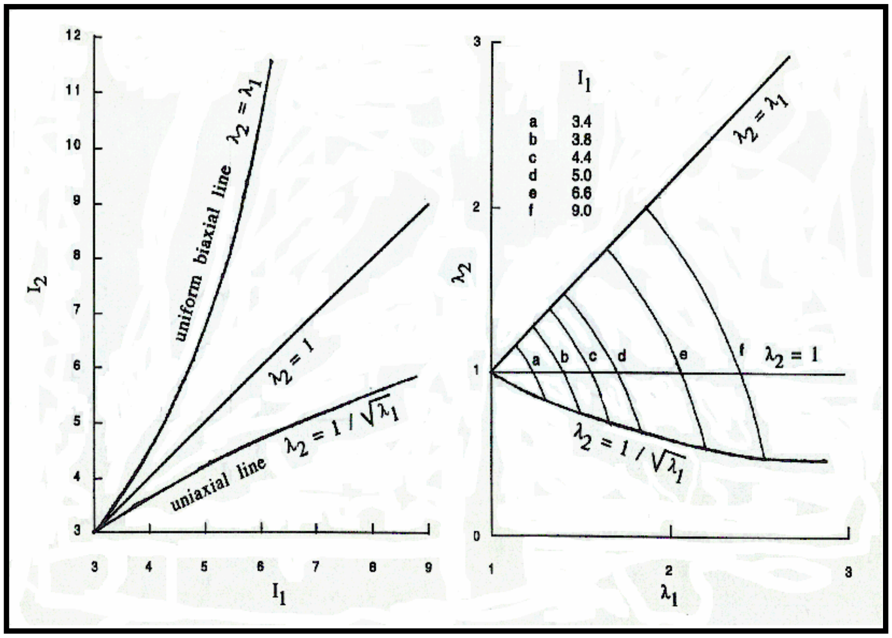

Figure 1. Domains of (I_1, I_2) (λ_1, λ_2) under the condition of incompressibility (adopted from Obata, Kawabata, and Kawai (1970))

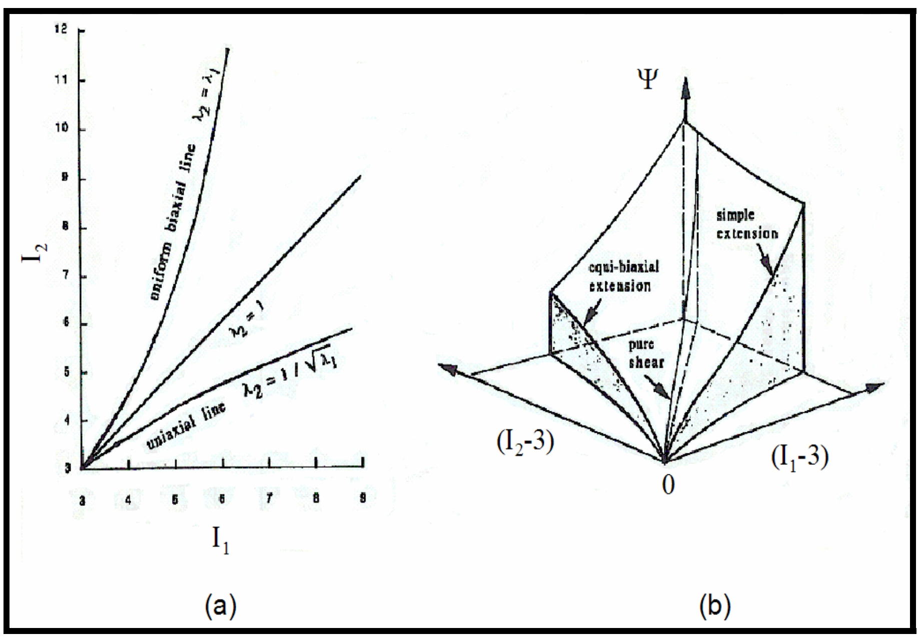

Figure 2. Map of the Helmholtz free-energy (strain energy) function $\Psi(I_1, I_2)$: (a) relation between I_1 and I_2 for incompressible material behavior; and (b) representation of the free-energy function as a function of I_1 I_2 (adopted from Morman (1985)).

Copyright © 2005 The ANSOL Corporation

REFERENCES

1. Morman, K. N., [1985]: "Rubber Viscoelasticity- A Review of Current Understanding", Proceedings of Second Symposium on Analysis and Design of Rubber Parts, London England, U.K., January 14-15 (1985), Sponsored by Battelle Institute, Ltd.
2. Treloar, L.R. C. [1975]: *The Physics of Rubber Elasticity*, Clarendon Press, Oxford.
3. Obata, Y., S. Kawabata, and H. Kawai [1970]: Mechanical properties of natural rubber vulcanizates in finite deformation, *Journal of Polymer Science*, Part A-2, **8**: 903-919.
4. Rivlin, R.S. [1948]: Large elastic deformations of isotropic materials, *Philosophical Transactions of The Royal Society of London* **A240**: 459-490.

ABOUT THE AUTHOR

Dr. Morman is the founder and President of The ANSOL Corporation, a consulting firm with clients in the tire and rubber industry and other industries concerned with the design and analysis of rubber components for durability. He has worked on the development of finite element analysis of rubber components and constitutive models of rubber at Ford Research Laboratory, Ford Motor Company for 28 years. He is the author or coauthor of over 30 journal articles on constitutive modeling of rubber and the finite element analysis of elastomer components and adhesively bonded joints. He has received the Henry Ford Technology Award from Ford Motor Company for his contributions to the development of a procedure for predicting door system and weather strip seal performance. In 1998 Dr. Morman was elected Fellow of The American Society of Mechanical Engineers (ASME) for his work in the development of constitutive models for adhesives and elastomeric materials. During the last two years before his retirement from Ford he was especially involved in the development and application of methods for predicting the effects of heat build-up and materials aging on the fatigue life of tires.

Dr. Morman was awarded the PhD degree in Engineering Mechanics from Columbia University in 1973.